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The Measurement of the Energy Spectral Density of Short Duration Impulsive Signals using the Digital Event Recorder Type 7502 and the Heterodyne Analyzer Type 2010

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THE MEASUREMENT OF THE ENERGY SPECTRAL DENSITY OF SHORT DURATION IMPULSIVE SIGNALS USING THE DIGITAL EVENT RECORDER TYPE 7502 AND THE HETERODYNE ANALYZER TYPE 2010

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INTRODUCTION

The concept of the Power Spectral Density of a periodic time function is well known, as are the techniques involved in its measurement. An equivalent measurement, using similar techniques, may be obtained from a single impulse. However, it is then more commonly referred to as the Energy Spectral Density. This Application Note looks first at some of the theory behind Energy Spectral Density, and then goes on to look at the practical side of its measurement for short duration impulsive signals using a Digital Event Recorder Type 7502 and a Heterodyne Analyzer Type 2010.

1. THEORY OF ENERGY SPECTRAL DENSITY

The energy, E, of a time varying function, v (t), is defined by the following equation as:

$$E = \int_{-\infty}^{\infty} |v(t)|^2 \, dt \tag{1}$$

A necessary consequence of this equation is that if the energy is non-infinite, v(t) can only exist for a finite period in time, i.e., it must be an impulsive or an aperiodic function. If v(t) exists for an infinite period in time, it becomes necessary to talk about its Power, rather than its energy. Its power, P, is then defined as:

$$P = \lim_{a \to \infty} \frac{1}{2a} \int_{-a}^{+a} |v(t)|^2 \, dt \tag{2}$$

The period of time 2a may be thought of as being the period of measurement. Hence, the power and the energy of a signal are fundamentally different. To have a finite and non zero power, a signal must be infinite in time, meaning that it has infinite energy, and to have a finite and non zero energy, a signal must be finite in time, meaning that it has zero power.

For periodic time functions, equation (2) reduces to:

$$P = \frac{1}{T} \int_{t'}^{t'+T} |v(t)|^2 \, dt \tag{3}$$

This equation leeds to Parseval's theorem, which relates the power of a function in the time domain to its form in the frequency domain.

It states that:

$$P = \sum_{k=-\infty}^{\infty} |V_k|^2 \tag{4}$$

Where V_k are the Fourier Coefficients. It is on this expression that the concept of Power Spectral Density is based. A similar kind of expression can be derived for a finite time function, this time relating its energy in the time domain to its form in the frequency domain. It is derived by first rewriting equation (1) as follows:

$$E = \int_{-\infty}^{\infty} v^*(t) \; v(t) \, dt \tag{5}$$

where v* (t) is the complex conjugate of v(t). In equation (5), v(t) may be replaced by its Fourier transform to obtain:

$$E = \int_{-\infty}^{\infty} v^*(t) \left[\int_{-\infty}^{\infty} V(f) \exp(j\,2\pi\, ft)\, df \right] dt \qquad (6)$$

The order of integration may then be revesed to give:

$$E = \int_{-\infty}^{\infty} V(f) \left[\int_{-\infty}^{\infty} v^*(t) \exp(j\,2\pi\, ft)\, dt \right] df \qquad (7)$$

The bracketted term in equation (7) is V* (f), the complex confugate of V(f). Hence:

$$E = \int_{-\infty}^{\infty} V(f)\ V^*(f)\, df = \int_{-\infty}^{\infty} |V(f)|^2\, df \qquad (8)$$

This is known as Rayleigh's theorem, and just as equation (4) is used as the basis for Power Spectral Density measurements, this equation is used as the basis for Energy Spectral Density measurements.

If we define W(f) to be the Energy Spectral Density of a finite time function, then we have:

$$E_{f_1 < f < f_2} = \int_{f_1}^{f_2} W(f)\, df \qquad (9)$$

and:

$$E = \int_{-\infty}^{\infty} W(f)\, df \qquad (10)$$

Comparison of equations (8) and (10) shows that the Energy Spectral Density, W(f), is in fact equal to the term $|V(f)|^2$. Hence, to obtain a measure of the Energy Spectral Density, all that is required is to be able to measure V(f).

2. MEASUREMENT OF ENERGY SPECTRAL DENSITY OF SHORT DURATION IMPULSIVE SIGNALS

2.1 Choice of Equipment

A suitable experimental set-up for the measurement of the Energy Spectral Density of a short duration impulsive signal is shown in Fig.1. It consists of a Digital Event Recorder Type 7502, a Heterodyne Analyzer Type 2010, and a Level Recorder Type 2305 or 2307. The 7502 enables the signal to be captured and reproduced repetitively with none of the problems normally associated with the use of a Type Recorder operating with a tape loop. An important feature of it is that the signal can be transformed upwards in frequency by a factor of up to 5000 on play back, which can permit a faster analysis. Whenever the capture and reproduction of short duration signals is required, this instrument is superior to the conventional Tape Recorder.

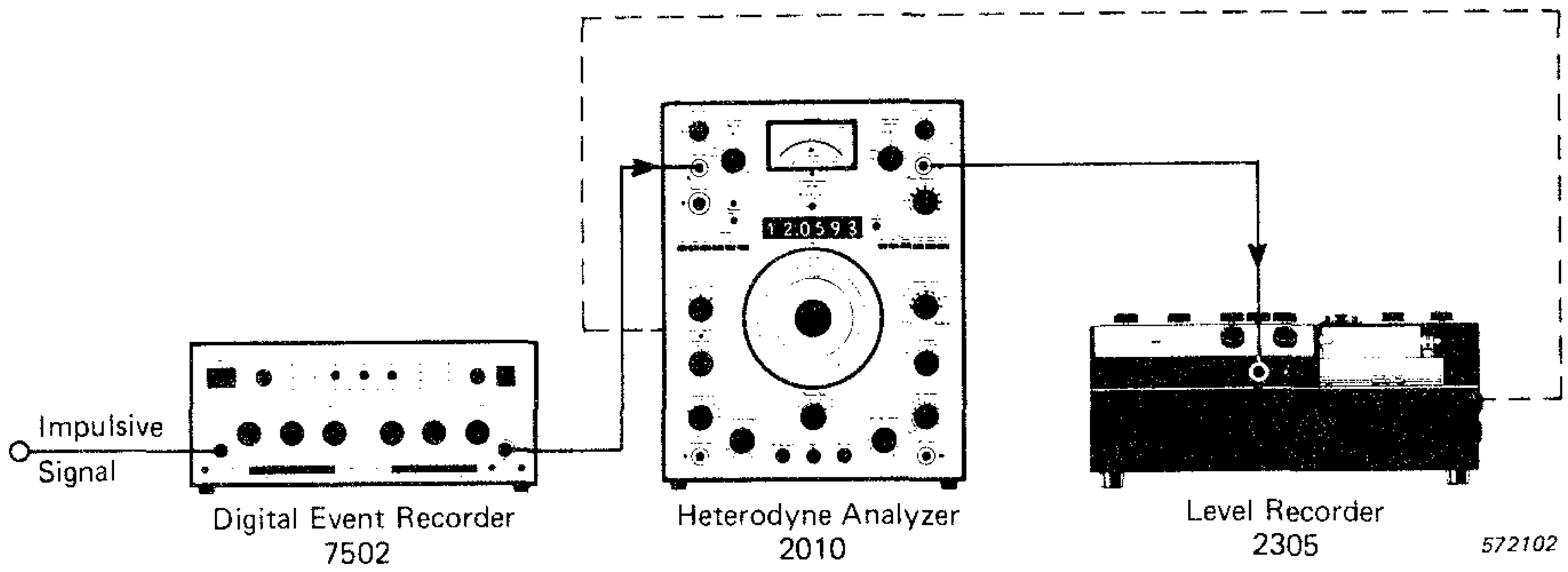


Fig. 1. Experimental set-up for the measurement of Energy Spectral Density

There are two methods by which a swept frequency analysis can be carried out. The first method, which here is called the "peak" method, uses a repetition time between the play backs which is long enough to ensure that each filter will only see the signal once, (i.e. the filter response has time to effectively decay to zero between one play back and the next). The filtered signal is passed through a peak rectifier before being recorded. In the second method, or "RMS" method, a shorter repetition time is used to ensure that each filter sees the signal several times, (i.e., the filter response never has time to decay to zero between play backs). The filtered signal is passed through an RMS rectifier being recorded. (These two methods should not be confused, since the treatment of the data that they produce differs). With the 7502, the range of repetition times available means that although either method may be used, it is probably better suited to the "RMS" method. Hence, the analyzer chosen is the Heterodyne Analyzer Type 2010, which has an RMS rectifier (but no peak rectifier). A further reason for using the 2010 is that it can be used for analysis up to 200 kHz, allowing the highest output sample rate on the 7502, i.e., 500 KS/s, to be used all the time. This means that the full range of frequency transformation is always available, which allows all analyses to be made at the fastest rate possible with this combination of equipment.

2.2 Practical Considerations

When setting up the system for an analysis, various factors have to be taken into account if optimum accuracy of analysis is to be achieved in the shortest possible time. These considerations are:

a) The 7502 should be set to play back at the fastest output sample rate available, i.e., 500 KS/s. This will allow the analysis to be carried out as quickly as possible. The lower limiting frequency of the analysis may then be taken as the frequency of the first line, i.e., the repetition frequency of the signal, and the upper limiting frequency will be 125 kHz.

b) The REPETITION DELAY Selector on the 7502 should be set such that the ratio between the signal duration and its repetition time is between 3 and 5. Other ratios may be used, but a lower one will mean a loss in detail in the spectrum, and a higher one will require a narrower bandwidth analyzing filter to pick up the individual lines in the spectrum, which will increase the analysis time.

c) The analyzing filter bandwidth on the 2010 should be set such that it is less than the reciprocal of the signal repetition time. (E.g., with a signal repetition time of 80 ms, a 10 Hz analyzing filter could be used). This is because the RMS method gives a line spectrum whose discrete frequencies are separated by the reciprocal of the repetition time, and for maximum possible accuracy, it is necessary to be able to discriminate between them. Hence, the filter must only cover one line at a time. Wider bandwidths may be used, but the readings must be corrected accordingly.

d) The averaging time set on the 2010 should be so as to minimise ripple on the Level Recorder trace. In most cases, an averaging time of three times the signal repetition time will be sufficient.

e) For a logrithmic frequency sweep, the PAPER SPEED on the Level Recorder should be set according to the following equation:

$$\text{maximum PAPER SPEED} = \frac{21.7 \times \Delta f}{f_{max} \times T} \quad \text{mm/sec} \tag{11}$$

where Δf is the bandwidth of the analyzing filter, f_{max} is the maximum frequency of interest, and T is the averaging time set on the 2010 or $2/\Delta f$, whichever is greater.

For a linear frequency sweep, the following equation should be used:

$$\text{maximum PAPER SPEED} = \frac{150 \times \Delta f}{2 \times T \times 10^a} \tag{12}$$

where a = 3 if the frequency scale is set to x 0.1
= 4 if the frequency scale is set to x 1
= 5 if the frequency scale is set to x 10

All other symbols have the same meaning as in equation (11).

The above equations only apply if Drive Shaft I of the Level Recorder is used. If Drive Shaft II is used they must be corrected accordingly.

f) The WRITING SPEED of the Level Recorder should be set high as possible to give stable writing conditions without overshoot.

g) The DC output of the 2010 should be used with the RECTIFIER RESPONSE of the Level Recorder set to DC.

2.3 Treatment of Data

It was found in Section 1 that the Energy Spectral Density of a finite time function was the square of the modulus of its Fourier Transform. What has been done so far is to take a finite time function, record it, and then repeat it to a spectrum analyzer. In doing this, the character of the function has been changed. It has become a periodic time function which is infinite in time. Hence, what has been recorded on the Level Recorder is not the Fourier transform of the function of interest, but the Fourier Series of that function when it is repeated at regular intervals. Thus, the question arises as to how we can convert this spectrum to the one of interest, i.e., the Fourier transform. What is required is some connection between the amplitude of a line in the Fourier Series, and the amplitude of the Fourier transform at the same frequency.

The solution to this problem may be found be considering a pulse, f(t), of duration T, which is repeated every T_R seconds. From the theory of Fourier Series we know that:

$$C_n = \frac{2}{T_R} \int_{-\infty}^{\infty} f(t) \exp(-j\, 2\pi\, nf_o\, t)\, dt \tag{13}$$

where f_o is the fundamental frequency, and C_n is the Fourier coefficient of the n^{th} harmonic. Since the pulse has duration T, this equation can be reduced to:

$$C_n = \frac{2}{T_R} \int_{-T/2}^{T/2} f(t) \exp(-j\, 2\pi\, nf_o t)\, dt \tag{14}$$

However, suppose we now treat f(t) as a single pulse. Its Fourier transform is then as follows:

$$F(f) = \int_{-\infty}^{\infty} f(t) \exp(-j2\pi\, ft)\, dt \tag{15}$$

It is again possible to change the limits of integration to give:

$$F(f) = \int_{-T/2}^{T/2} f(t) \exp(-j2\pi\, ft)\, dt \tag{16}$$

Now, if we compare equations 14 and 16, it can be seen that where $f = nf_o$:

$$C_n = \frac{2\ F(nf_o)}{T_R} = \frac{2\ F(f)}{T_R} \tag{17}$$

Hence:

$$F(f) = \frac{C_n \cdot T_R}{2} \tag{18}$$

Thus, to obtain the value of the Fourier transform at $F = nf_o$, all that is required is to multiply C_n by $T_r/2$. However, in the measurement made using the 2010, the RMS value of C_n was measured, and not the peak value. Hence, equation (18) must be modified by a factor of $\sqrt{2}$ (since we can assume a near sinusoidal signal after filtering) to give:

$$F(f) = \frac{T_R\ C_n}{\sqrt{2}} \tag{19}$$

Thus, to obtain the required Fourier transform from the measured spectrum, all that is required is to multiply each line by a factor of $T_R/\sqrt{2}$. The envelope of the resulting spectrum is the Fourier transform, which is then squared to give the Energy Spectral Density.

2.4 Accuracy of Results

There are two main sources of error in this measurement. The first is the actual measurement of the spectrum itself, and the second is the manipulation of the measured spectrum to give the Energy Spectral Density. The errors introduced from the first part have already been considered elsewhere. Suffice it to say that an error of 1 dB in the measured level will produce an error of approximately 20% in the Energy Spectral Density. The errors introduced in the second part however, are entirely dependent on how the manipulation to obtain the Energy Spectral Density is carried out, and hence cannot be defined. It is thus not possible to define an overall accuracy for this method. What can be done though, is to see practically the degree of accuracy that can be obtained by analyzing impulses of known energy, and then comparing the result obtained from integrating their Energy Spectral Densities with this value.

The accuracy of the method was checked with six different types of impulse, varying from a rectangular impulse to a tone burst. Each impulse was recorded on the 7502, and then analyzed following the guide lines of Section 2.2 regarding the settings of the instrument controls. After the completion of the analysis, a trace of the impulse in the time domain was obtained with a direct play back from the 7502 to the Level Recorder. This trace was then used as the basis for calculating the energy in the pulse, which was then compared with the value obtained by integrating the Energy Spectral Density.

The integral of the Energy Spectral Density was obtained as follows: the lines in the spectrum given by the 2010 were multiplied by $T_R/\sqrt{2}$, as required by equation (19), and then squared. All lines whose amplitude was more than 20 dB below the peak were ignored. What resulted was hence a series of points in the Energy Spectral Density curve, separated by $1/T_R$. The integration was then completed by multiplying each point by $1/T_R$, taking the sum, and then doubling it (to allow for the fact that the integration should be from minus infinity to plus infinity). Mathematically, the process can be expressed as follows:

$$E = \frac{2}{T_R} \sum \left(\frac{T_R \ C_n}{\sqrt{2}} \right)^2$$

where C_n is the amplitude of the n^{th} line. This was then taken as the energy in the impulse. The error in this value, assuming the energy obtained from the trace in the time domain to be correct, is plotted in Table 1 for each type of impulse used.

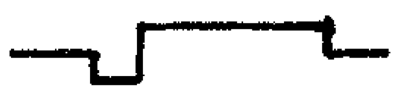
Impulse	Error
N-wave	6.5%
rectangular pulse	2%
trangular pulse	4%
tone burst	6%
	4.5%
	2%

Table 1

2.5 Practical Measurement of Energy Spectral Density

The final stage of this investigation was to record and analyze a practical impulsive signal. Due to its complexity and its broadband spectrum, the type of impulsive signal chosen for analysis was Punch Press Noise.

Before the start of the analysis, the system was calibrated using a 1 kHz 94 dB (1 Pa.) calibration tone. This was recorded in the 7502 and then played back with the same repetition ratio as would be used with the Punch Press Noise itself. The output from the 2010 was then set to a convenient point on the Level Recorder such that all future measurements could be referenced to this point. When this procedure had been completed, the punch press noise was recorded and analyzed following the guide lines given in Section 2.2.

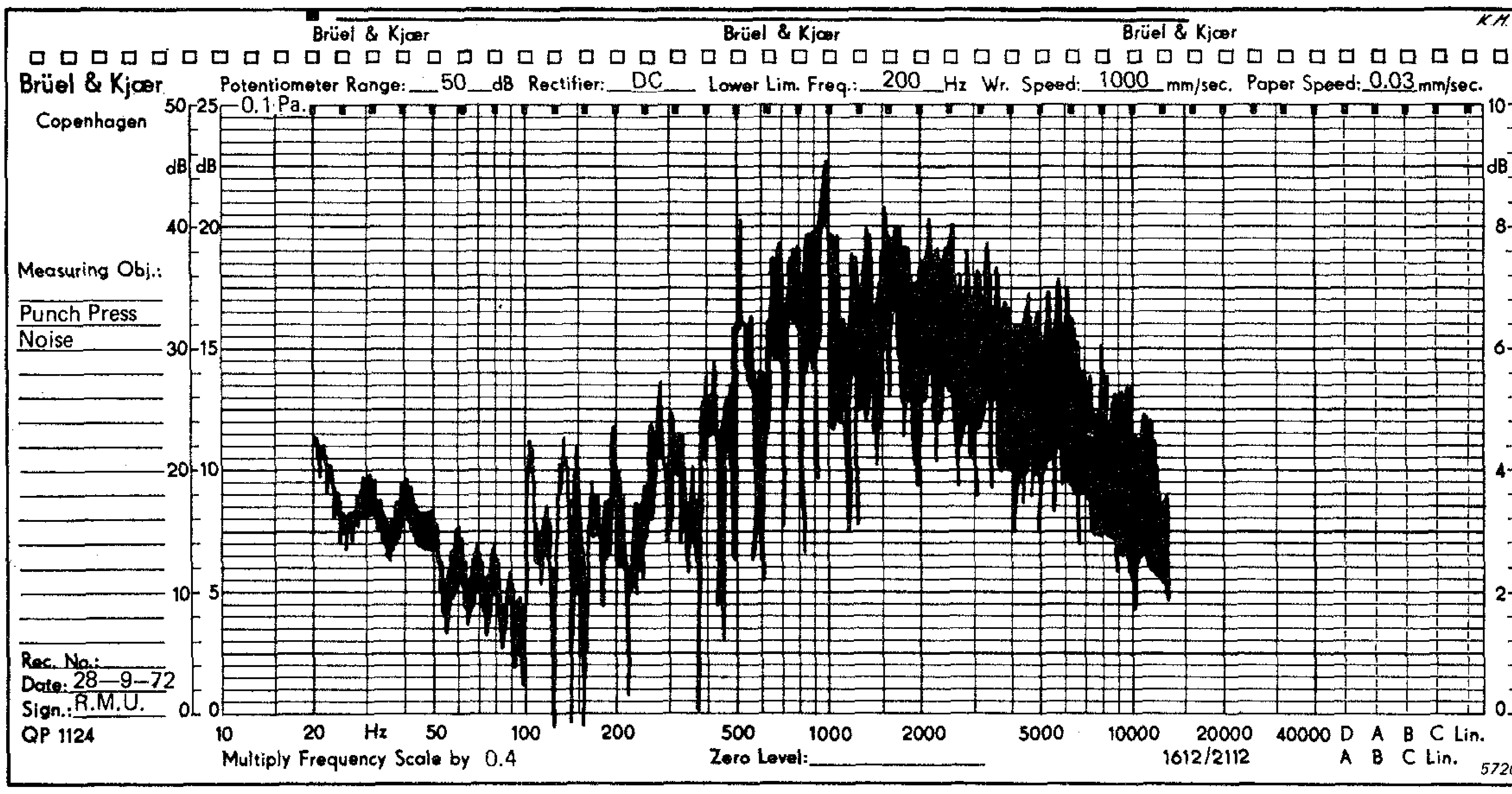


Fig.2. Frequency Analysis of Punch Press Noise

An analysis of the signal is given in Fig.2. Since the signal was recorded at 20 KS/s and analyzed at 500 KS/s, a frequency transformation ratio of 25 is displayed. The bandwidth of the analyzing filter was 10 Hz, and the averaging time was 0.3 seconds. This particular analysis was used to see where the main signal content lay. It was then repeated using Drive Shaft II of the Level Recorder to drive the 2010, so that an expanded frequency scale could be used on the paper. The Frequency Marking facility of the 2010 was used to provide the frequency scale. A portion of the expanded analysis (from about 8 kHz to about 10 kHz) around the point where the peak value occurs is shown in Fig.3. This peak occurs at approximately 9.3 kHz on the analysis, which corresponds to 372 Hz on the true frequency scale, and has an energy of $\sim 1.1 \times 10^{-5}$ $(Pa/Hz)^2$. The expanded analysis was then used to find the total energy in the impulse, which corresponded to $\sim 2.8\ Pa^2/Hz$.

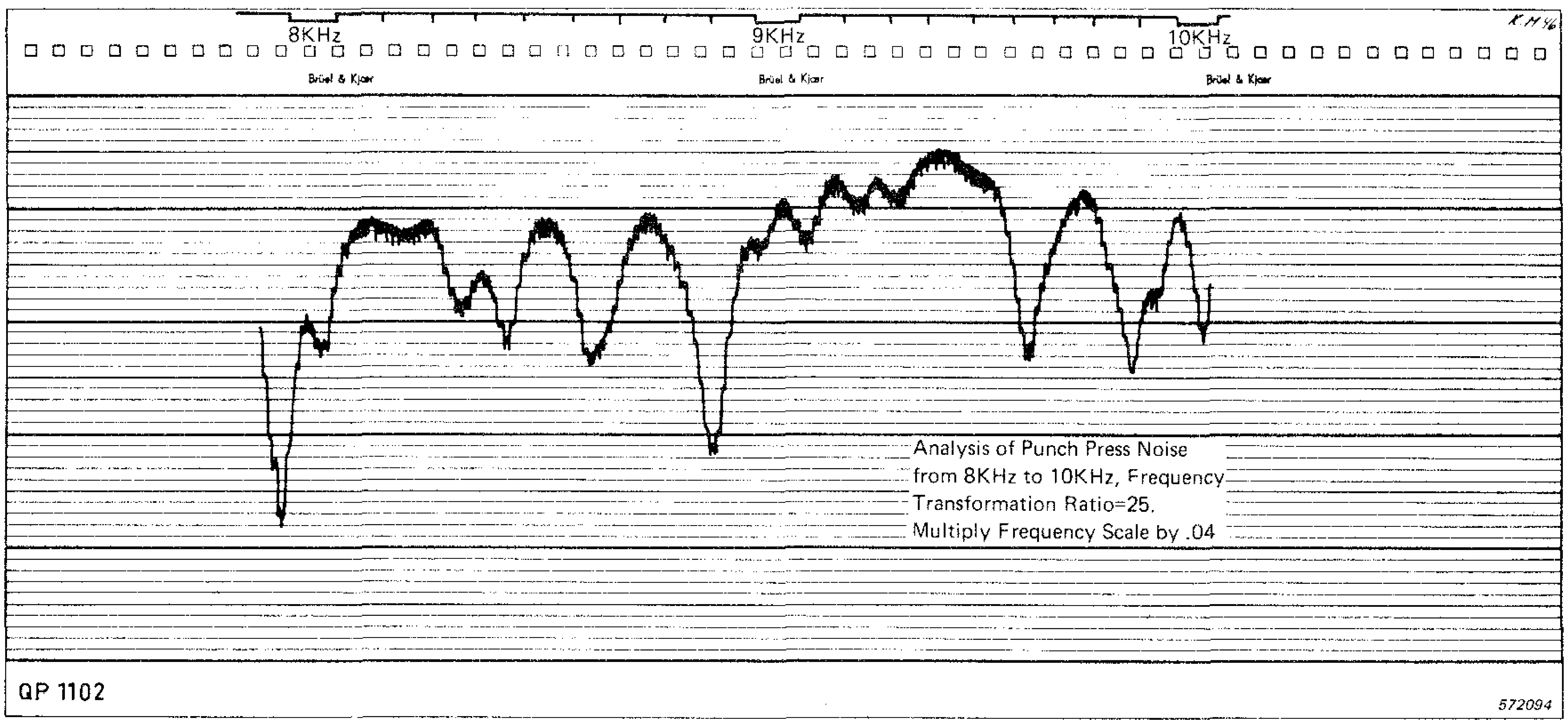


Fig.3. Frequency Analysis of Punch Press Noise with expanded frequency scale

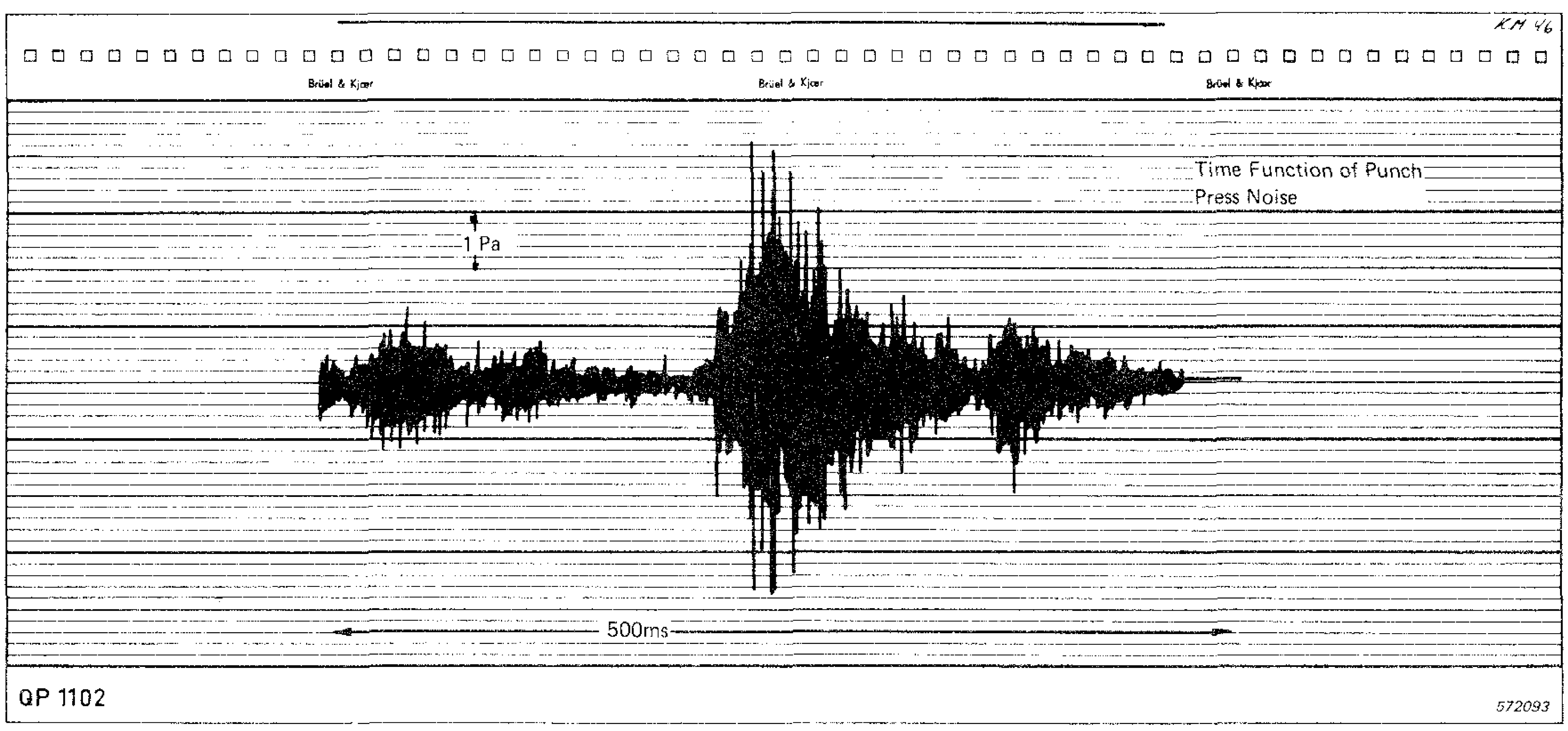


Fig.4. Play Back of Punch Press Noise in the Time Domain

A play back of the impulse in the time domain is given in Fig.4. When this is used with a play back of the calibration tone under the same conditions, the peak instantaneous values of the impulse may be measured.

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